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NOTE

On the Order of a Generalized Hexagon Admitting an Ovoid or Spread

Alan Offer

c/Goiri 7, 2 Izda, 28039 Madrid, Spain
E-mail: alan_offer@hotmail.com

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a spread only if $s = t$. © 2001 Elsevier Science

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A generalized hexagon of order (s, t) is a geometry of points and lines in which each line is incident with exactly $s+1$ points, each point is incident with exactly $t+1$ lines, and whose incidence graph has diameter 6 and girth 12. A generalized hexagon is just said to have order s when $s = t$. Further details, including introductory information, can be found in [2, Section 7.3], [3] and [4].

Let Γ be a generalized hexagon of order (s, t). Given two elements u and v of Γ, the *distance* $\delta(u, v)$ between them is the distance between them in the incidence graph of Γ. The elements u and v are *opposite* if $\delta(u, v) = 6$. An *ovoid* of Γ is then a set $\mathcal{O}$ of mutually opposite points such that for each element u of Γ there is a point $v \in \mathcal{O}$ such that $\delta(u, v) \leqslant 3$. *Spreads* are defined dually, and although we consider only ovoids here, all of that which follows holds dually for spreads. Notice further that we will say that two points of Γ are *collinear* if they are at distance 2 from each other.

Let $\mathcal{O}$ be a set of mutually opposite points in the generalized hexagon Γ. It is shown in [4, Proposition 7.2.3] that the set $\mathcal{O}$ is an ovoid if and only if $|\mathcal{O}| = (1+s)(1+st+s^2t^2)/(1+s+st)$. In particular, if Γ has order s then $\mathcal{O}$ is an ovoid if and only if $|\mathcal{O}| = 1+s^3$. Let us see how this is proved there. Consider a point $p \in \mathcal{O}$. Since there are $t+1$ lines through p and on each of

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these lines there are an additional s points, there are $s(t+1)$ points collinear with p. Given another point $q \in \mathcal{O}$, the points collinear with it are all distinct from those collinear with p since p and q are opposite. It follows that there are $s(t+1)\,|\mathcal{O}|$ points that are collinear with some point of $\mathcal{O}$. Including the points of $\mathcal{O}$ as well, the total number of points u for which there is some point $v \in \mathcal{O}$ with $\delta(u, v) \leqslant 3$ is then $(1+s+st)\,|\mathcal{O}|$. Finally, by the definition of ovoids, the set $\mathcal{O}$ is an ovoid precisely when this value is equal to the total number of points in Γ, which is $(1+s)(1+st+s^2t^2)$ (see, for example, [2, 7.3.7]). The result follows.

Suppose now that $\mathcal{O}$ is an ovoid of Γ. Let p be a point of $\mathcal{O}$ and let L be a line incident with p. Since there are a further s points on L, a further t lines through each of these points (that is, in addition to L), and then a further s points on each of these, the set A of points q for which $\delta(p, q) = 4$ and $\delta(L, q) = 3$ contains s^2t elements. Since $\mathcal{O}$ is an ovoid, each of the points in A is collinear with exactly one point of $\mathcal{O} \setminus \{p\}$. Conversely, each point of $\mathcal{O} \setminus \{p\}$ is collinear with some point of A. Furthermore, no two distinct points of A are collinear with a common point of $\mathcal{O}$ as such an event would result in there being a triangle, quadrangle or pentagon in Γ, contrary to its incidence graph having girth 12. Thus $|\mathcal{O}| = 1+s^2t$. This value for $|\mathcal{O}|$ has appeared before, being mentioned, for instance, in [1]. However, what seems hitherto not to have been done is to equate this with the value from the previous paragraph. Doing so, we have

$$(1+s)(1+st+s^2t^2) = (1+s+st)(1+s^2t)$$

which upon expansion becomes

$$1+st+s^2t^2+s+s^2t+s^3t^2 = 1+s+st+s^2t+s^3t+s^3t^2.$$

Cancelling the common terms, all that remains is $s^2t^2 = s^3t$ which readily reduces to $s = t$. Thus we have demonstrated the following.

THEOREM. *Let Γ be a generalized hexagon of order (s, t) and let $\mathcal{O}$ be a set of mutually opposite points in Γ. Then $\mathcal{O}$ is an ovoid of Γ if and only if $s = t$ and $|\mathcal{O}| = s^3+1$. Dually, a set $\mathcal{S}$ of mutually opposite lines is a spread if and only if $s = t$ and $|\mathcal{S}| = s^3+1$.*

Notice that it follows that a generalized hexagon of order (s, s^3) or (s^3, s) cannot have an ovoid (nor a spread), a result which has previously been obtained by noting that $|\mathcal{O}| = (1+s)(1+st+s^2t^2)/(1+s+st)$ must be an integer and then considering the two cases separately (see [4, 7.2.4]).

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